

# Self-dual gravity is completely integrable

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## Abstract

We discover multi-Hamiltonian structure of complex Monge-Ampère equation (*CMA*) set in a real first-order two-component form. Therefore, by Magri's theorem this is a completely integrable system in four real dimensions. We start with Lagrangian and Hamiltonian densities and obtain a symplectic form and the Hamiltonian operator that determines the Dirac bracket. We have calculated all point symmetries of two-component *CMA* system and Hamiltonians of the symmetry flows. We have found two new real recursion operators for symmetries which commute with the operator of a symmetry condition on solutions of the *CMA* system. These operators form two Lax pairs for the two-component system. The recursion operators, applied to the first Hamiltonian operator, generate infinitely many real Hamiltonian structures. We show how to construct an infinite hierarchy of higher commuting flows together with the corresponding infinite chain of their Hamiltonians.

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## 1 Introduction

In earlier paper [1] we presented complex multi-Hamiltonian structure of Plebański's second heavenly equation [2], which by Magri's theorem [3] proves that it is a completely integrable system in four complex dimensions. We expect that Plebański's first heavenly equation also admits multi-Hamiltonian structure, since these two equations, governing Ricci-flat metrics with (anti-)self-dual Riemann curvature 2-form, are related by Legendre transformation

of corresponding heavenly tetrads [2]. However, since both Plebański's equations are complex, their solutions are potentials of the complex metrics that satisfy Einstein equations in complex four-dimensional spaces. In the case of complex Monge-Ampère equation (*CMA*), that governs (anti-)self-dual gravity in real four-dimensional spaces with either Euclidean or ultra-hyperbolic signature, we have an additional condition that symplectic, Hamiltonian, and recursion operators all should be real. Furthermore, the transformation between the two heavenly equations cannot be applied to transform the second heavenly equation to *CMA* because the latter equation is real and therefore the multi-Hamiltonian structure of *CMA* cannot be obtained by transforming the multi-Hamiltonian structure of the second heavenly equation given in [1]. Therefore, in this paper we analyze the complex Monge-Ampère equation independently of our previous work and obtain real recursion operators for symmetries and real multi-Hamiltonian structures of *CMA*.

In section 2 we start with the complex Monge-Ampère equation in a two-component first-order evolutionary form with the Lagrangian that is appropriate for Hamiltonian formulation. In section 3 we present a symplectic structure and Hamiltonian structure of this *CMA* system. In section 4 we transform the Hamiltonian density and Hamiltonian operator to real variables and introduce a convenient notation needed later to arrive at a compact form of recursion operators and higher Hamiltonian operators. In section 5 we derive a symmetry condition, that determines symmetries of *CMA* system, in a two-component form and real variables, using our new notation. We have calculated all point symmetries of the *CMA* system and Hamiltonians of the symmetry flows that yield conservation laws for the *CMA* system. In section 6 we obtain two new real recursion operators for symmetries which commute with an operator of the symmetry condition on solutions of the *CMA* system. Moreover, these two couples of operators form two Lax pairs for the two-component system. The recursion operators, being applied to the first Hamiltonian operator, yield further Hamiltonian structures and bi-Hamiltonian representations of the *CMA* system. Repeating this procedure for the second Hamiltonian operator, we could generate infinitely many Hamiltonian structures of the *CMA* system. This multi-Hamiltonian structure of the *CMA* system proves its complete integrability in the sense of Magri and hence complete integrability of the (anti-)self-dual gravity in four real dimensions with either Euclidean or ultra-hyperbolic signature. In section 7 we construct an infinite hierarchy of higher flows and show the way of calculating a corresponding infinite chain of higher Hamiltonians.

## 2 Complex Monge-Ampère equation in first-order evolutionary form and its Lagrangian

Four-dimensional hyper-Kähler metrics

$$ds^2 = u_{1\bar{1}}dz^1d\bar{z}^1 + u_{1\bar{2}}dz^1d\bar{z}^2 + u_{2\bar{1}}dz^2d\bar{z}^1 + u_{2\bar{2}}dz^2d\bar{z}^2 \quad (2.1)$$

satisfy Einstein field equations with either Euclidean or ultra-hyperbolic signature, if the Kähler potential  $u$  satisfies elliptic or hyperbolic complex Monge-Ampère equation

$$u_{1\bar{1}}u_{2\bar{2}} - u_{1\bar{2}}u_{2\bar{1}} = \varepsilon \quad (2.2)$$

with  $\varepsilon = \pm 1$  respectively [2]. Here  $u$  is a real-valued function of the two complex variables  $z^1, z^2$  and their conjugates  $\bar{z}^1, \bar{z}^2$ , the subscripts denoting partial derivatives with respect to these variables. Such metrics are Ricci-flat and have (anti-)self-dual curvature.

In order to discuss the Hamiltonian structure of *CMA* (2.2), we shall replace the complex conjugate pair of variables  $z^1, \bar{z}^1$  by the real time variable  $t = 2\Re z^1$  and the real space variable  $x = 2\Im z^1$  and change the notation for the second complex variable  $z^2 = w$ . Then (2.2) becomes

$$(u_{tt} + u_{xx})u_{w\bar{w}} - u_{tw}u_{t\bar{w}} - u_{xw}u_{x\bar{w}} + i(u_{tw}u_{x\bar{w}} - u_{xw}u_{t\bar{w}}) = \varepsilon. \quad (2.3)$$

Now we can express (2.3) as a pair of first-order nonlinear evolution equations by introducing an auxiliary dependent variable  $v = u_t$

$$\begin{cases} u_t = v \\ v_t = -u_{xx} + \frac{1}{u_{w\bar{w}}} \left( v_w v_{\bar{w}} + u_{xw} u_{x\bar{w}} + i(v_{\bar{w}} u_{xw} - v_w u_{x\bar{w}}) + \varepsilon \right), \end{cases} \quad (2.4)$$

so that finally (2.2) is set in a two-component form. For the sake of brevity we shall henceforth refer to (2.4) as *CMA* system.

The Lagrangian density for the original form (2.2) of the complex Monge-Ampère equation was suggested in [4]

$$L = \frac{1}{6} [u_1 u_{\bar{1}} u_{2\bar{2}} + u_2 u_{\bar{2}} u_{1\bar{1}} - u_1 u_{\bar{2}} u_{2\bar{1}} - u_2 u_{\bar{1}} u_{1\bar{2}}] + \varepsilon u, \quad (2.5)$$

but this must be cast into a form suitable for passing to a Hamiltonian. This requires that the form of a Lagrangian should be appropriate for applying

Dirac's theory of constraints [5]. We choose the Lagrangian density for the first-order *CMA* system (2.4) to be degenerate, that is, linear in the time derivative of unknown  $u_t$  and with no  $v_t$ :

$$L = \frac{1}{6} \{ (u_x^2 - 3v^2) u_{w\bar{w}} + u_w u_{\bar{w}} u_{xx} - u_x (u_w u_{x\bar{w}} + u_{\bar{w}} u_{xw}) + u_t (2i(u_w u_{x\bar{w}} - u_{\bar{w}} u_{xw}) + 6v u_{w\bar{w}}) \} + \varepsilon u \quad (2.6)$$

which, after substituting  $v = u_t$ , coincides with our original Lagrangian (2.5) up to a total divergence.

### 3 Symplectic and Hamiltonian structures

Since the Lagrangian density (2.6) is linear in  $u_t$  and has no  $v_t$ , the canonical momenta

$$\begin{aligned} \pi_u &= \frac{\partial L}{\partial u_t} = \frac{i}{3} (u_w u_{x\bar{w}} - u_{\bar{w}} u_{xw}) + v u_{w\bar{w}} \\ \pi_v &= \frac{\partial L}{\partial v_t} = 0 \end{aligned} \quad (3.1)$$

cannot be inverted for the velocities  $u_t$  and  $v_t$  and so the Lagrangian is degenerate. Therefore, according to the Dirac's theory [5], we impose them as constraints

$$\begin{aligned} \phi_u &= \pi_u + \frac{i}{3} (u_{\bar{w}} u_{xw} - u_w u_{x\bar{w}}) - v u_{w\bar{w}} = 0 \\ \phi_v &= \pi_v = 0 \end{aligned} \quad (3.2)$$

and calculate the Poisson brackets of the constraints (more details of the procedure were given in [1])

$$K_{ik} = [\phi_i(x, w, \bar{w}), \phi_k(x', w', \bar{w}')] \quad (3.3)$$

collecting results in a  $2 \times 2$  matrix form, where the subscripts run from 1 to 2 with 1 and 2 corresponding to  $u$  and  $v$  respectively. This yields the symplectic operator  $K$  that is the inverse of the Hamiltonian operator  $J_0$ :

$$K = \begin{pmatrix} (v_{\bar{w}} - i u_{x\bar{w}}) D_w + (v_w + i u_{xw}) D_{\bar{w}} + v_{w\bar{w}} & -u_{w\bar{w}} \\ u_{w\bar{w}} & 0 \end{pmatrix} \quad (3.4)$$

as an explicitly skew-symmetric local operator. A symplectic 2-form is a volume integral  $\Omega = \int_V \omega dx dw d\bar{w}$  of the density

$$\omega = \frac{1}{2} du^i \wedge K_{ij} du^j = \frac{1}{2} (v_{\bar{w}} - i u_{x\bar{w}}) du \wedge du_w + \frac{1}{2} (v_w + i u_{xw}) du \wedge du_{\bar{w}} + u_{w\bar{w}} dv \wedge du \quad (3.5)$$

where  $u^1 = u$  and  $u^2 = v$ . In  $\omega$ , under the sign of the volume integral, we can neglect all the terms that are either total derivatives or total divergencies due to suitable boundary conditions on the boundary surface of the volume.

For the exterior differential of this 2-form we obtain

$$d\omega = -i du_x \wedge du_w \wedge du_{\bar{w}} = -(i/3) \left( D_x (du \wedge du_w \wedge du_{\bar{w}}) + D_w (du_x \wedge du \wedge du_{\bar{w}}) + D_{\bar{w}} (du_x \wedge du_w \wedge du) \right) \Longleftrightarrow 0 \quad (3.6)$$

that is, a total divergence which is equivalent to zero, so that the 2-form  $\Omega$  is closed and hence symplectic. The Hamiltonian operator  $J_0$  is obtained by inverting  $K$  in (3.4)

$$J_0 = \begin{pmatrix} 0 & \frac{1}{u_{w\bar{w}}} \\ -\frac{1}{u_{w\bar{w}}} & \frac{v_{\bar{w}} - i u_{x\bar{w}}}{2u_{w\bar{w}}^2} D_w + D_w \frac{v_{\bar{w}} - i u_{x\bar{w}}}{2u_{w\bar{w}}^2} + \frac{v_w + i u_{xw}}{2u_{w\bar{w}}^2} D_{\bar{w}} + D_{\bar{w}} \frac{v_w + i u_{xw}}{2u_{w\bar{w}}^2} \end{pmatrix} \quad (3.7)$$

that is explicitly skew-symmetric. It satisfies the Jacobi identity due to (3.6).

The Hamiltonian density is

$$H_1 = \pi_u u_t + \pi_v v_t - L$$

with the result

$$H_1 = \frac{1}{6} \left[ (3v^2 - u_x^2) u_{w\bar{w}} - u_w u_{\bar{w}} u_{xx} + u_x (u_{\bar{w}} u_{xw} + u_w u_{x\bar{w}}) \right] - \varepsilon u. \quad (3.8)$$

*CMA* system can now be written in the Hamiltonian form with the Hamiltonian density  $H_1$  defined by (3.8)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} \quad (3.9)$$

where  $\delta_u$  and  $\delta_v$  are Euler-Lagrange operators [6] with respect to  $u$  and  $v$  applied to the Hamiltonian density  $H_1$  (they correspond to variational derivatives of the Hamiltonian functional  $\int_V H_1 dV$ ).

## 4 Transformation to real variables

In the case of *CMA*, that governs (anti-)self-dual gravity with either Euclidean or ultra-hyperbolic signature, we have an additional condition that all the objects in the theory, in particular a recursion operator, should be real. Therefore, we transform the Hamiltonian density together with the symplectic and Hamiltonian operators to the real variables  $y = 2\Re w$  and  $z = 2\Im w$ . The Hamiltonian density in the real variables becomes

$$H_1 = \frac{1}{6} \left[ (3v^2 - u_x^2) \Delta(u) - (u_y^2 + u_z^2) u_{xx} + 2u_x(u_y u_{xy} + u_z u_{xz}) \right] - \varepsilon u.$$

where  $\Delta(u) = u_{yy} + u_{zz}$ , which simplifies after cancelling terms that are total derivatives to

$$H_1 = \frac{1}{2} [v^2 \Delta(u) - u_{xx}(u_y^2 + u_z^2)] - \varepsilon u. \quad (4.1)$$

The transformation of the Hamiltonian operator  $J_0$  in (3.7) yields

$$J_0 = \begin{pmatrix} 0 & \frac{1}{a} \\ -\frac{1}{a} & \frac{1}{a^2}(cD_y - bD_z) + (D_y c - D_z b) \frac{1}{a^2} \end{pmatrix} \quad (4.2)$$

where we introduce the notation

$$a = \Delta(u), \quad b = u_{xy} - v_z, \quad c = v_y + u_{xz}, \quad Q = \frac{b^2 + c^2 + \varepsilon}{a} \quad (4.3)$$

that we will use from now on throughout the paper, with  $D_y, D_z$  designating operators of total derivatives with respect to  $y, z$  respectively and  $\Delta = D_y^2 + D_z^2$  is the two-dimensional Laplace operator.

The symplectic operator (3.4) in the real variables becomes

$$K = J_0^{-1} = \begin{pmatrix} cD_y - bD_z + D_y c - D_z b & -a \\ a & 0 \end{pmatrix} \quad (4.4)$$

in an explicitly skew-symmetric form.

*CMA* system (2.4) in the real variables becomes

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = \begin{pmatrix} v \\ Q - u_{xx} \end{pmatrix} \quad (4.5)$$

or  $u_t = v$ ,  $v_t = Q - u_{xx}$ .

The four-dimensional hyper-Kähler metrics (2.1) in the real variables in the notation (4.3) become

$$ds^2 = \frac{1}{4} [Q(dt^2 + dx^2) + a(dy^2 + dz^2)] + \frac{1}{2} [c(dtdy + dx dz) - b(dtdz - dx dy)]. \quad (4.6)$$

The metrics (4.6) satisfy Einstein field equations with either Euclidean or ultra-hyperbolic signature, if the two-component potential  $(u, v)$  in the definitions (4.3) of  $a, b, c$ , and  $Q$  satisfies the Hamiltonian *CMA* system (4.5) with  $\varepsilon = +1$  or  $\varepsilon = -1$  respectively. These metrics are again Ricci-flat and have (anti-)self-dual curvature.

## 5 Symmetries and integrals of motion

Now, consider Lie group of transformations of the system (4.5) in the evolutionary form, when only dependent variables are transformed, and let  $\tau$  be the group parameter. Then Lie equations read

$$u_\tau = \varphi, \quad v_\tau = \psi \quad (5.1)$$

where  $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$  is a two-component symmetry characteristic of the system (4.5). The differential compatibility conditions of equations (4.5) and (5.1) in the form  $u_{t\tau} - u_{\tau t} = 0$  and  $v_{t\tau} - v_{\tau t} = 0$  result in the linear matrix equation

$$\mathcal{A}(\Phi) = 0 \quad (5.2)$$

where  $\mathcal{A}$  is the Frechét derivative of the flow (4.5)

$$\mathcal{A} = \begin{pmatrix} D_t & -1 \\ D_x^2 - \frac{2}{a}(cD_z + bD_y)D_x + \frac{Q}{a}\Delta, & D_t - \frac{2}{a}(cD_y - bD_z) \end{pmatrix} \quad (5.3)$$

where the first row of (5.2) yields  $\varphi_t = \psi$ .

Using the software packages LIEPDE and CRACK by T. Wolf [7], run under REDUCE 3.8, we have calculated all point symmetries of *CMA* system (4.5), a class of solutions of the matrix equation (5.2). We list their generators and two-component symmetry characteristics [6], the latter denoted by  $\varphi^u, \varphi^v$

$$X_1 = t\partial_t + x\partial_x + u\partial_u, \quad \varphi_1^u = u - tv - xu_x, \quad \varphi_1^v = t(u_{xx} - Q) - xv_x$$

$$\begin{aligned}
X_2 &= z\partial_y - y\partial_z, \quad \varphi_2^u = yu_z - zu_y, \quad \varphi_2^v = yv_z - zv_y \\
X_3 &= \partial_z, \quad \varphi_3^u = u_z, \quad \varphi_3^v = v_z \\
X_4 &= \partial_y, \quad \varphi_4^u = u_y, \quad \varphi_4^v = v_y \\
X_5 &= y\partial_y + z\partial_z + u\partial_u + v\partial_v, \quad \varphi_5^u = u - yu_y - zu_z, \quad \varphi_5^v = v - yv_y - zv_z \\
X_\alpha &= \alpha(t, x, y, z)\partial_u + \alpha_t(t, x, y, z)\partial_v, \quad \varphi_\alpha^u = \alpha, \quad \varphi_\alpha^v = \alpha_t \\
X_\beta &= \beta_z(y, z)\partial_x - \beta_y(y, z)\partial_t, \quad \varphi_\beta^u = \beta_y v - \beta_z u_x, \quad \varphi_\beta^v = \beta_y(Q - u_{xx}) - \beta_z v_x
\end{aligned} \tag{5.4}$$

where  $\alpha(t, x, y, z)$  is an arbitrary smooth solution of the equations

$$\Delta(\alpha) = 0, \quad \alpha_{tt} + \alpha_{xx} = 0, \quad \alpha_{tz} - \alpha_{xy} = 0, \quad \alpha_{ty} + \alpha_{xz} = 0 \tag{5.5}$$

whereas  $\beta(y, z)$  satisfies the two-dimensional Laplace equation  $\Delta(\beta) = 0$ .

We shall find the integrals of motion generating the point symmetries that serve as Hamiltonians of the symmetry flows

$$\begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = \begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix} \tag{5.6}$$

where the symmetry group parameter  $\tau$  plays the role of time for the symmetry flow (5.6) and  $\mathcal{H} = \int_{-\infty}^{+\infty} H dx dy dz$  is an integral of the motion along the flow (4.5), with the conserved density  $H$ , that generates the symmetry with the two-component characteristic  $\varphi_u, \varphi_v$ . The second equality in (5.6) is the Hamiltonian form of Noether's theorem that gives a relation between symmetries and integrals.

We choose here the Poisson structure determined by our first Hamiltonian operator  $J_0$  since we know its inverse  $K$  given by (4.4) which is used in the inverse Noether theorem

$$\begin{pmatrix} \delta_u H \\ \delta_v H \end{pmatrix} = K \begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix} \tag{5.7}$$

determining conserved densities  $H$  corresponding to known symmetry characteristics  $\varphi_u, \varphi_v$ .

Using (5.7), we reconstruct the Hamiltonians of the flows (5.6) for all variational point symmetries in (5.4). For the scaling symmetries generated by  $X_1$  and  $X_5$ , Hamiltonians do not exist and so they are not variational symmetries. For the rotational symmetry generated by  $X_2$ , the Hamiltonian is

$$H_2 = v(yu_z - zu_y)\Delta(u) - u_x[2(zu_y + yu_z)u_{yz} + u_y^2 + u_z^2]. \tag{5.8}$$



For the translational symmetries generated by  $X_3$  and  $X_4$ , the corresponding Hamiltonians  $H_3$  and  $H_4$  are

$$\begin{aligned} H_3 &= vu_z\Delta(u) + \frac{2}{3}u_x(u_yu_{zz} - u_zu_{yz}) \\ H_4 &= vu_y\Delta(u) + \frac{2}{3}u_x(u_yu_{yz} - u_zu_{yy}). \end{aligned} \quad (5.9)$$

For the infinite Lie pseudogroups generated by  $X_\alpha$  and  $X_\beta$ , the Hamiltonians are

$$H_\alpha = \alpha v\Delta(u) + \frac{1}{2}\alpha_t(u_y^2 + u_z^2) + \alpha(u_zu_{xy} - u_yu_{xz}) \quad (5.10)$$

and

$$H_\beta = \left(\frac{\beta_y}{2}v^2 - \beta_zu_xv\right)\Delta(u) - \frac{\beta_y}{2}u_{xx}(u_y^2 + u_z^2) + \frac{1}{2}u_x^2(\beta_{yy}u_y + \beta_{yz}u_z) - \varepsilon\beta_yu. \quad (5.11)$$

In particular, the Hamiltonian of time translations  $X_{\beta=y} = -\partial_t$ , that is  $H_{\beta=y} = H_1$ , coincides with the Hamiltonian (4.1) of *CMA* flow. For translations in  $x$ ,  $X_{\beta=z} = \partial_x$ , the Hamiltonian is  $H_{\beta=z} = -u_xv\Delta(u)$ . For a simple example of the symmetry  $X_\alpha$ ,  $X_{\alpha=z} = z\partial_u$ , the Hamiltonian is  $H_{\alpha=z} = zv\Delta(u) + u_xu_y$ , which coincides with the Hamiltonian  $H_0$  (6.7) from an infinite chain of Hamiltonians for a hierarchy of higher commuting flows in section 6.

All of these Hamiltonians of the symmetry flows are conserved densities of the *CMA* flow (4.5).

## 6 Recursion operators and bi-Hamiltonian representations of *CMA* system

Complex recursion operators for symmetries of the heavenly equations of Plebański were introduced in the papers of Dunajski and Mason [8, 9]. We have used them in our method of partner symmetries for obtaining non-invariant solutions of complex Monge-Ampère equation [10, 11] and second heavenly equation of Plebański [11] and, in a two-component form, for generating multi-Hamiltonian structure of Plebański's second heavenly equation in [1]. However, for *CMA* we have an additional condition that the equation and its symmetries are real and hence recursion operators should also be real.

This condition leads to a couple of real recursion operators in a  $2 \times 2$  matrix form. The first one is

$$R_1 = \begin{pmatrix} 0 & 0 \\ QD_z - cD_x & b \end{pmatrix} \quad (6.1)$$

$$+ \Delta^{-1} \begin{pmatrix} D_y(-aD_x + bD_y + cD_z) + D_z(cD_y - bD_z) & -D_z a \\ D_x[D_y(cD_y - bD_z) + D_z(aD_x - bD_y - cD_z)] & -D_x D_y a \end{pmatrix}$$

where  $\Delta^{-1}$  means operator multiplication, and the second recursion operator reads

$$R_2 = \begin{pmatrix} 0 & 0 \\ bD_x - cD_y & c \end{pmatrix} \quad (6.2)$$

$$+ \Delta^{-1} \begin{pmatrix} D_y(bD_z - cD_y) + D_z(-aD_x + bD_y + cD_z) & D_y a \\ D_x[D_y(-aD_x + bD_y + cD_z) + D_z(cD_y - bD_z)] & -D_x D_z a \end{pmatrix}$$

Straightforward, though cumbersome, calculations show that the operators  $R_1$  and  $R_2$  commute with the operator  $\mathcal{A}$  (5.3) of the symmetry condition (5.2) on solutions of equations (4.5) and therefore they are indeed recursion operators for symmetries of the *CMA* system. This means that if  $(\tilde{\varphi}, \tilde{\psi})$  is obtained by transforming a two-component symmetry characteristic  $(\varphi, \psi)$  of the system (4.5) by the operator  $R_1$  or  $R_2$

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} = R_i \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \quad (6.3)$$

where  $i = 1, 2$ , then  $(\tilde{\varphi}, \tilde{\psi})$  is also a symmetry characteristic of (4.5).

Moreover, vanishing of the commutators  $[R_i, \mathcal{A}]$ , computed without using the equations of motion (4.5), reproduces the *CMA* system (4.5) and hence the operators  $R_i$  and  $\mathcal{A}$  form two real Lax pairs for the two-component system. Indeed, introducing a short-hand notation  $F = u_t - v$ ,  $G = v_t + u_{xx} - Q$ ,  $\Phi = F_{xy} - G_z$  and  $\chi = F_{xz} + G_y$ , we rewrite the *CMA* system in the form  $F = 0$ ,  $G = 0$ , so that  $\Phi = 0$  and  $\chi = 0$  on its solutions, and the first commutator reads

$$[R_1, \mathcal{A}] = \begin{pmatrix} 0 & 0 \\ \frac{1}{a}[Q\Delta(F) - 2(b\Phi + c\chi)]D_z & -\Phi \end{pmatrix} + \Delta^{-1} \times \quad (6.4)$$

$$\left( \begin{array}{cc} \Phi(D_z^2 - D_y^2) - 2\chi D_y D_z & D_z \Delta(F) \\ + D_y \Delta(F) D_x - \Delta(F_x) D_y - \Delta(G) D_z, & \\ 2(D_z \chi D_x + D_x \Phi D_y) D_z - D_z \Delta(F) D_x^2 & \\ + \chi_x (D_z^2 - D_y^2) + [2\chi_z - \Delta(G)] D_x D_y & D_x D_y \Delta(F) \\ + \Delta(G_y) D_x - \Delta(G_x) D_y & \end{array} \right)$$

and we have a similar expression for  $[R_2, \mathcal{A}]$ . Thus,  $[R_i, \mathcal{A}] = 0$  implies  $F = 0$ ,  $G = 0$ , that is, the *CMA* system (4.5).

These real Lax pairs are formed by the recursion operators for symmetries and operator  $\mathcal{A}$  of the symmetry condition and so they are Lax pairs of the Olver-Ibragimov-Shabat type [14,15], which is different from the complex Lax pairs suggested by Mason and Newman [12,13] and Dunajski and Mason [8,9] and those that we used in [10,11] in relation to partner symmetries, even if we set our new Lax pairs in one-component forms. Furthermore, the commutator of the complex recursion operator of Mason-Dunajski with the operator of the symmetry condition, in a one-component form, reproduces the symmetry condition and not the original equation *CMA* [10].

By the theorem of Magri, given a Hamiltonian operator  $J$  and a recursion operator  $R$ ,  $RJ$  is also a Hamiltonian operator [3]. Thereby, acting by the recursion operator  $R_1$  on the first Hamiltonian operator  $J_0$  (3.7), we obtain the second Hamiltonian operator

$$J_1 = R_1 J_0 = \Delta^{-1} \left( \begin{array}{cc} D_z & -D_x D_y \\ D_x D_y & D_x^2 D_z \end{array} \right) + \quad (6.5)$$

$$\left( \begin{array}{cc} 0 & \frac{b}{a} \\ -\frac{b}{a} & \frac{c}{a^2} (b D_y - a D_x) + (D_y b - D_x a) \frac{c}{a^2} + \frac{Q_-}{2a} D_z + D_z \frac{Q_-}{2a} \end{array} \right)$$

that is explicitly skew-symmetric. Here  $Q_- = (c^2 - b^2 + \varepsilon)/a$ . The proof of the Jacobi identity for  $J_1$  is lengthy but can be somewhat facilitated by using Olver's criterion in terms of functional multi-vectors [6].

Similarly, acting by the recursion operator  $R_2$  on the Hamiltonian operator  $J_0$ , we obtain a Hamiltonian operator that is another companion for  $J_0$ :

$$J^1 = R_2 J_0 = \Delta^{-1} \left( \begin{array}{cc} D_y & D_x D_z \\ -D_x D_z & D_x^2 D_y \end{array} \right) + \quad (6.6)$$

$$\begin{pmatrix} 0 & -\frac{c}{a} \\ \frac{c}{a} & \frac{b}{a^2}(cD_z - aD_x) + (D_z c - D_x a)\frac{b}{a^2} + \frac{Q^-}{2a}D_y + D_y\frac{Q^-}{2a} \end{pmatrix}$$

that is also explicitly skew-symmetric and  $Q^- = (b^2 - c^2 + \varepsilon)/a$ . The Jacobi identity for  $J^1$  was also proved by using Olver's criterion in terms of functional multi-vectors [6].

The flow (4.5) can be generated by the Hamiltonian operator  $J_1$  from the Hamiltonian density

$$H_0 = zv\Delta(u) + u_x u_y \quad (6.7)$$

so that CMA in the two-component form (4.5) admits two Hamiltonian representations

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \quad (6.8)$$

and thus this is a *bi-Hamiltonian system*.

The same flow (4.5) can also be generated by the Hamiltonian operator  $J^1$  from the Hamiltonian density

$$H^0 = yv\Delta(u) - u_x u_z \quad (6.9)$$

which yields another bi-Hamiltonian representation of the CMA system (4.5)

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J^1 \begin{pmatrix} \delta_u H^0 \\ \delta_v H^0 \end{pmatrix}. \quad (6.10)$$

Repeating this procedure  $n$  times, we obtain a *multi-Hamiltonian representation* of the CMA system with the Hamiltonian operators  $J_n = R_1^n J_0$ ,  $J^n = R_2^n J_0$ ,  $J_m^{n-m} = R_1^m R_2^{n-m} J_0$  ( $m = 1, 2, \dots, n-1$ ) and corresponding Hamiltonian densities. This procedure will be considered in more detail in the next section for the operator  $R_1$ . Multi-Hamiltonian structure of the CMA system proves its complete integrability in the sense of Magri and hence the complete integrability of the (anti-)self-dual gravity in four real dimensions with either Euclidean or ultra-hyperbolic signature.

A totally different recursion operator for the (anti-)self-dual gravity in complex Einstein spaces was obtained much earlier by Strachan [16] by using a Legendre transformed version of the first heavenly equation, that was derived by Grant [17]. This recursion operator can be factorized which suggests a bi-Hamiltonian structure of the resulting evolutionary equation, though

that was not completely proved. However, the evolutionary equation and the related Hamiltonian structures are expressed in complex variables, with the complex "time"  $t$  in particular, and with a complex unknown. Therefore, the corresponding metric will not correspond to anti-self-dual gravity in real Einstein spaces with the Euclidean signature  $(++++)$ . Furthermore, the Poisson bracket contains the unusual operator  $\partial_t^{-1}$  that could be avoided in a two-component formulation.

## 7 Infinite hierarchy of higher flows

The operators  $J_0$  and  $J_1$  are compatible Hamiltonian operators, i.e. they form a Poisson pencil. This means that every linear combination  $C_0 J_0 + C_1 J_1$  with constant coefficients  $C_0$  and  $C_1$  satisfies the Jacobi identity. This can be more easily verified by using the Olver's criterion in terms of functional multi-vectors though the calculation is still very lengthy. We know from the work of Fuchssteiner and Fokas [18] (see also the survey [19] and references therein) that if a recursion operator has a factorized form, as in our case  $R_1 = J_1 J_0^{-1} \equiv J_1 K$ , and the factors  $J_0$  and  $J_1$  are compatible Hamiltonian operators, then  $R_1$  is hereditary (Nijenhuis) recursion operator, i.e. it generates an Abelian symmetry algebra out of commuting symmetry generators. Moreover, Hermitian conjugate hereditary recursion operator  $R_1^\dagger = J_0^{-1} J_1 = K J_1$ , acting on the vector of variational derivatives of an integral of the flow, yields a vector of variational derivatives of some other integral of this flow. Then (6.8) implies that  $R_1^\dagger$  generates the Hamiltonian density  $H_1$  from  $H_0$ :

$$R_1^\dagger \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_0^{-1} J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} \quad (7.1)$$

where  $R_1^\dagger$  is defined by

$$R_1^\dagger = \begin{pmatrix} 0 & D_x c - D_z Q \\ 0 & b \end{pmatrix} + \begin{pmatrix} (-D_x a + D_y b + D_z c) D_y & [(D_z b - D_y c) D_y + (-D_x a + D_y b + D_z c) D_z] D_x \\ + (D_y c - D_z b) D_z & a D_x D_y \end{pmatrix} \Delta^{-1}. \quad (7.2)$$

The first higher flow of the hierarchy is generated by  $J_1$  acting on the vector of variational derivatives of  $H_1$

$$\begin{pmatrix} u_{t_1} \\ v_{t_1} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} \quad (7.3)$$

where  $t_1$  is the time variable of the higher flow. This flow is nonlocal and the right-hand side of (7.3) is too lengthy to be presented here explicitly.

Now we could generate the next Hamiltonian  $H_2$  of the hierarchy of commuting flows by applying  $R_1^\dagger$  to the vector of variational derivatives of  $H_1$ :

$$R_1^\dagger \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = K J_1 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix}. \quad (7.4)$$

Therefore, the second higher flow in the hierarchy has a bi-Hamiltonian representation

$$\begin{pmatrix} u_{t_2} \\ v_{t_2} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} = J_1 R_1^\dagger \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} \quad (7.5)$$

where the third Hamiltonian operator  $J_2$  is generated by acting with  $R_1$  on  $J_1$ :  $J_1 R_1^\dagger = J_1 K J_1 = R_1 J_1 = J_2$ . Acting by  $J_2$  on the variational derivatives of  $H_0$ , we obtain the relations

$$\begin{aligned} J_2 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} &= J_1 R_1^\dagger \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} \\ &= J_0 R_1^\dagger \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} \end{aligned} \quad (7.6)$$

where we have used that  $J_1 = J_0(KJ_1) = J_0 R_1^\dagger$ . From (7.6) we obtain three-Hamiltonian representation of the first higher flow

$$\begin{pmatrix} u_{t_1} \\ v_{t_1} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix}. \quad (7.7)$$

We could also construct the Hamiltonian  $H_{-1}$  such that  $H_0$  is generated from  $H_{-1}$  by  $R_1^\dagger$

$$R_1^\dagger \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix} = J_0^{-1} J_1 \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix} = \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \quad (7.8)$$

that implies a bi-Hamiltonian representation for the zeroth flow

$$\begin{pmatrix} u_{t_0} \\ v_{t_0} \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix}. \quad (7.9)$$

Further we obtain

$$J_2 \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix} = J_1 R_1^\dagger \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \quad (7.10)$$

and the bi-Hamiltonian representation (6.8) of the original two-component *CMA* flow becomes a three-Hamiltonian representation of this flow

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u H_{-1} \\ \delta_v H_{-1} \end{pmatrix}. \quad (7.11)$$

We could still continue by applying  $R_1^\dagger$  to the vector of variational derivatives of  $H_2$  to generate the next Hamiltonian  $H_3$

$$R_1^\dagger \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} = K J_1 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} = \begin{pmatrix} \delta_u H_3 \\ \delta_v H_3 \end{pmatrix} \quad (7.12)$$

and obtain a bi-Hamiltonian representation for the next higher flow

$$\begin{pmatrix} u_{t_3} \\ v_{t_3} \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_3 \\ \delta_v H_3 \end{pmatrix} = R_1 J_1 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} = J_2 \begin{pmatrix} \delta_u H_2 \\ \delta_v H_2 \end{pmatrix} \quad (7.13)$$

where we have used (7.12) and the relation  $J_1 K J_1 = R_1 J_1 = J_2$ , and so on.

## 8 Conclusion

Our starting point was the symplectic and Hamiltonian structure of the complex Monge-Ampère equation, set into a two-component evolutionary form. We have calculated all point symmetries of the *CMA* system and also, using inverse Noether theorem, Hamiltonians of the flows for all variational symmetries. These Hamiltonians yield conservation laws for the *CMA* flow. We have found two real  $2 \times 2$  matrix recursion operators  $R_1$  and  $R_2$  for symmetries that commute with the operator  $\mathcal{A}$  of the symmetry condition and hence map any symmetry of the *CMA* system again into a symmetry. The operators  $R_1$  and  $R_2$  together with  $\mathcal{A}$  form two Lax pairs for the

two-component  $CMA$  system. Acting on the first Hamiltonian operator by each recursion operator, we obtain two new Hamiltonian operators according to Magri's theorem [3] and two bi-Hamiltonian representations of the complex Monge-Ampère equation in the two-component form. Repeating this action, we could generate an infinite number of Hamiltonian operators and hence construct a multi-Hamiltonian representation of the  $CMA$  system. We show how to construct an infinite hierarchy of higher commuting flows together with the corresponding infinite chain of their Hamiltonians by using a Hermitian conjugate recursion operator. In particular, we arrive at three-Hamiltonian representations for both  $CMA$  flow and the first higher flow and bi-Hamiltonian representations for the zeroth flow and second higher flow. The results of this paper prove complete integrability of the (anti-)self-dual gravity in four real dimensions in the sense of Magri (a multi-Hamiltonian representation).

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